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CONSTANT SIZE CONTROL IN STABILITY ESTIMATES UNDER SOME RESOLVENT CONDITIONS

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The paper deals with the question of stability of a discrete semigroup under certain resolvent conditions on its generator. The main objective is to examine the behavior of the stability constants as functions of the constant in the original resolvent estimate.

1. Preliminaries and the main results[2]**.** Beyond all manner of doubt, one of the crucial points in the analysis of discretizations of differential equations is that of stability. At the same time, for stability itself it is most important to settle satisfactorily the question of power boundedness of linear operators. The results in this direction seem to be far from complete even in the Hilbert space framework (see, e.g., Nagy & Foiaş [18]) while, for practical needs, many authors recently turned to showing power boundedness in Banach norms under some reasonable conditions. Over the last years, the problem of power boundedness in Banach space settings has been extensively examined in the literature via using certain resolvent conditions on the operator in question, among others, the so-called Kreiss and Tadmor conditions and related ones. For such results in the finite-dimensional case, we mention the work of Kreiss [12], Morton [17], Miller & Strang [16], Tadmor [26, 27], Le Veque & Trefethen [13], and Spijker [24]. As concerns the infinite-dimensional case, we refer to our work [1, 2, 3, 5], El-Fallah & Ransford [8], Kalton *et al* [10], Lubich & Nevanlinna [14], Lyubich [15], Nagy & Zemánek [19], Nevanlinna [20, 21, 22], Spijker & Straetemans [25], and Vitse [28, 29] (see also the references therein).

Let X be a Banach space and let $\mathcal{B}(X)$ be the algebra of all linear bounded operators on X. Given $\mathfrak{L} \in \mathcal{B}(X)$, we denote by $\sigma(\mathfrak{L})$ and $\|\mathfrak{L}\|$ the spectrum and norm of $\mathfrak{L}$, respectively. For our subsequent needs, we denote as well, for $z \in \mathbb{C}$, $r \geq 0$, and $\psi \in [0, \pi]$,

$$\mathcal{D}(z; r) = \{\lambda \in \mathbb{C} : |\lambda - z| \leq r\} \quad \text{and} \quad \Sigma_\psi = \{\lambda \in \mathbb{C} : |\arg \lambda| \leq \psi\}.$$

Also, given a set $V \subset \mathbb{C}$, we use ∂V and $V^\odot$ to denote the boundary of V and the set $\overline{\mathbb{C} \setminus V}$, respectively. Apart from everything else, for $0 < a < 1$ and $0 < \alpha \leq \arcsin a$, we define $\Upsilon(a, \alpha)$ as a subset of $\mathbb{C}$ given by

$$\Upsilon(a, \alpha) = \mathcal{D}(1; a) \cup \big(\Sigma_\alpha \cap \mathcal{D}(0; d)\big),$$

where

$$d = \cos\alpha - (a^2 - \sin^2\alpha)^{1/2} \tag{1}$$

(see Figure 1).

Let further $\mathfrak{U} \in \mathcal{B}(X)$. The family of operators $\mathfrak{U}^n$, $n = 0, 1, \ldots$, is then called a discrete semigroup. In what follows of concern will be the problem of estimation of the discrete semigroup $\mathfrak{U}^n$ in the norm $\|\cdot\|$ as well as in some related weighted norms, under some reasonable assumptions on the localization of $\sigma(\mathfrak{U})$ and on the behavior of the resolvent $(\lambda I - \mathfrak{U})^{-1}$ outside of the spectrum. Note that even having stated the fact that $\sup_{n\geq 0} \|\mathfrak{U}^n\| < \infty$ is most important for qualitative analysis of stability. However, for practical applications one needs to control the size of the constants in the corresponding stability estimates, taking into account that such constants may be of big size or even be growing functions of some parameters of the method. In this paper we therefore examine the behavior of the stability constants as functions of the constant in the original resolvent estimate, as such we consider two, close to each other in some sense, resolvent conditions, which will be specified below.

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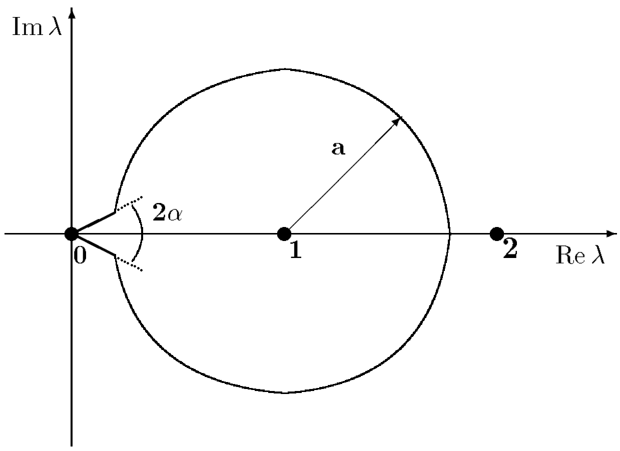

Fig. 1. The set $\Upsilon(a, \alpha)$

Throughout the paper we shall denote by C and c generic constants, subject to $C \ge 0$ and $c > 0$, whose sizes will be unessential for our analysis. They may depend on other constants appearing in the context but they never depend on M occurring in the below estimate (2).

Now we state the resolvent conditions accepted in the present paper. In our subsequent consideration we actually deal not only with power boundedness but, as already mentioned above, we intend to show some weighted estimates related to that expressing the power boundedness. In view of this, given a discrete semigroup $\mathfrak{U}^n$, $n \ge 0$, it will be convenient to pose the starting resolvent conditions in terms of the operator $\mathfrak{A} = I - \mathfrak{U}$, which is called the generator of the discrete semigroup $\mathfrak{U}^n$. More precisely, we accept the following requirement on $\mathfrak{A}$, with a set V suitably chosen and with some $M \ge 1$,

$$\left\|(\lambda I - \mathfrak{A})^{-1}\right\| \le M|\lambda|^{-1} \quad \text{for all} \quad \lambda \notin \operatorname{Int} V. \tag{2}$$

In fact, we admit of only two possibilities for choosing V. More precisely, we state the following two hypotheses:

H1: $\mathfrak{A}$ satisfies (2) with $V = \Upsilon(a, \alpha)$,

for some fixed $0 < a < 1$ and $0 < \alpha \le \arcsin a$, and

H2: $\mathfrak{A}$ satisfies (2) with $V = \mathcal{D}(1; 1)$.

As concerns the constant M in (2), we suppose that its size may vary but, at the same time, it remains under control. The main aim of the present paper will be therefore to trace the dependence of the constants in the final stability estimates on M.

We remark that the results stated in [1][3] are based on making use of (2), with $V \subset \mathcal{D}(1; 1)$ taken in the form of driving wheel whose jags have a sharp contact with the circle $\partial\mathcal{D}(1; 1)$ (see the figure in [1]). The main resolvent condition in [1] is therefore a slight generalization of the above hypothesis **H1** in the case when $M = C$. It has been shown in [4] that such a situation is really natural for the so-called parabolic case. It seems, however, that Soviet Mathematics Doklady is hardly popular among the West readers. The main assertion on power boundedness from [1], more precisely, its particular case under hypothesis **H1** with $M = C$ has been repeatedly restated afterwards. Moreover, it was, in fact, the most essential component of the approaches used in Lyubich [15] and Nagy and Zemánek [19] while the original work [1] seems to have never been even referenced. The results of [1] remained unnoticed in spite of the fact that they had given an impact for a big series of works on stability of both autonomous and nonautonomous equations[4], problems with splitting operator, ill-posed problems, problems with generalized input data, problems on nonuniform grids, etc. (see, e.g., the references in [6]) — most of these developments appeared in the West literature or, at least, translated into English. Note

[3] The corresponding proofs are given in [2, 3, 5].

[4] In the stability analysis of discretizations of nonautonomous equations one has, in fact, to estimate products of linear operators, which are noncommutative, in general. Obviously, this is a more difficult problem than that of estimation of powers of linear operators. It is worth noting that the consideration in [1] covers the nonautonomous case as well.

that the dependence of the stability constants on M was not examined in [1], as if it were accepted that $M = C$. Our first result here (see Theorem 1.1 below) asserts that the stability constant has a linear growth with M. In order to show that, we use, in fact, our old techniques, which have been previously presented only in Russian editions, except for a Spanish report [6].

Note further that accepting hypothesis **H2** is equivalent to the Tadmor resolvent condition (cf. Tadmor [27]). This condition is also related to the so-called Ritt resolvent condition introduced in Ritt [23]. It is known that both conditions are, in fact, equivalent in some sense (see remarks in Borovykh, Drissi & Spijker [7] and Vitse [28]) but, assuming the Ritt condition, one can only show that **H1** holds in such a way that the constant M cannot be specified. It turns out that using hypothesis **H2** or, equivalently, the Tadmor resolvent condition, the stability constants have a more involved behavior, in comparison with the case when **H1** is accepted. At the same time, in our opinion, hypothesis **H2** cannot be thought of as more natural than hypothesis **H1**, and, perhaps, **H2** is rather of theoretical interest. In fact, as it follows by analytic continuation, **H2** implies **H1** if with a bigger constant M_1 in place of M. This means that the discrete problem in question is naturally parabolic. On the other hand, as it has been shown, for instance, in [4], hypothesis **H1** holds naturally after reasonable discretization of parabolic problems and should be therefore thought of as original, not as a produce of **H2**, with some $M_0 < M$ in place of M. Nevertheless, **H2** is used here for the reason that it is extensively discussed in the literature and they give rise to the question in which way the size of the constant M appears in stability estimates. By the way, both Ritt [23] and Tadmor [27], themselves, have not really shown the uniform stability of the discrete semigroup with respect to n, based on the conditions suggested by them, while Lyubich [15] and Nagy & Zemanek [19] have been able to do this, using the above-mentioned equivalence between the above hypotheses **H1** and **H2**. In [15] and [19] they do not, however, trace the constants of stability (as functions of M); this problem is just settled in the present paper (see Theorem 1.2 below), via analytic continuation of the resolvent. At the same time it has been shown in Borovykh, Drissi, and Spijker [7] (cf. also El-Fallah & Ransford [8] and Vitse [29]), under a condition, which is equivalent to stating hypothesis **H2**, that $\|\mathfrak{U}^n\| \leq CM^2$, while Theorem 1.2 below gives

$$\|\mathfrak{U}^n\| \leq CM\log(1+M).$$

Also, actually, under the same restriction, Yuan [30] has found an estimate, which is equivalent to the fact that

$$\|\mathfrak{A}\,\mathfrak{U}^n\| \leq CM^3(n+1)^{-1},$$

while it follows from Theorem 1.2 below that

$$\|\mathfrak{A}\,\mathfrak{U}^n\| \leq CM^2\log(1+M)\,(n+1)^{-1}.$$

It is worth noting that the present advancement, in comparison with the results of [7, 30], has become possible due to our techniques, which allow us to work with the resolvent outside of sets whose configurations are like those considered in [1]. On top of all this, Theorem 1.2 gives an estimation of the quantity $\|\mathfrak{A}^\xi\mathfrak{U}^n\|$ for fractional values of $\xi \geq 0$.

Now we state the main results of this paper. In our consideration it is assumed that we are given a discrete semigroup $\mathfrak{U}^n$, $n \geq 0$, and $\mathfrak{A}$ stands for its generator, that is $\mathfrak{A} = I - \mathfrak{U}$.

Theorem 1.1. *Let $\mathfrak{A}$ satisfy hypothesis* **H1** *with some fixed $a \in (0,1)$ and $\alpha \in (0, \arcsin a]$. Then, for any fixed $\xi \geq 0$, we have for all $n = 0, 1, \ldots$,*

$$\|\mathfrak{A}^\xi\mathfrak{U}^n\| \leq CM(n+1)^{-\xi}. \tag{3}$$

Theorem 1.2. *Let $\mathfrak{A}$ satisfy hypothesis* **H2**. *Then, for any fixed $\xi \geq 0$, we have for all $n = 0, 1, \ldots$,*

$$\|\mathfrak{A}^\xi\mathfrak{U}^n\| \leq CK(M)\,(n+1)^{-\xi}, \tag{4}$$

where, with $[\xi]$ the integral part of ξ,

$$K(M) = \begin{cases} M^{1+\xi}\log(1+M) & \text{if} \quad \xi = [\xi], \\ M^{1-[\xi]+2\xi} & \text{if} \quad [\xi] < \xi \leq [\xi] + \dfrac{1}{2}, \\ M^{2+[\xi]} & \text{if} \quad [\xi] + \dfrac{1}{2} < \xi < [\xi] + 1. \end{cases}$$

The proofs of these assertions are given in the next section.

Note that the above estimates (3) and (4) have counterparts in holomorphic semigroup theory, for a closed operator A satisfying the resolvent condition, with some fixed $\varphi \in (0, \pi/2]$,

$$\left\|(\lambda I - A)^{-1}\right\| \le M|\lambda|^{-1} \text{ for } \lambda \in \Sigma_\varphi^\odot,$$

which implies, in particular, that A generates a holomorphic bounded semigroup e^{-tA}. More precisely, it is then possible to estimate the quantity $\|A^\xi e^{-At}\|$, tracing the dependence on M, provided that one distinguishes between the cases when $0 < \varphi < \pi/2$ and $\varphi = \pi/2$ (for the result in the case $M = C$, see, e.g., Komatsu [11, Theorem 12.2]).

2. Proofs.

Proof of Theorem 1.1. Assume first that $\xi = 0$. Let then Γ^n be a positively oriented contour given by

$$\partial\Big(\mathcal{D}\big(0; d(n+1)^{-1}\big) \cup \Upsilon(a, \alpha)\Big), \quad n = 0, 1, \dots,$$

where d is given by (1). By the Dunford–Taylor operator calculus formula, we have for all $n = 0, 1, \dots,$

$$\mathfrak{U}^n = (2\pi i)^{-1} \int\limits_{\Gamma^n} (1-\lambda)^n (\lambda I - \mathfrak{A})^{-1}\, d\lambda. \tag{5}$$

It follows from (5), after a simple estimation, that for $n = 0, 1, \dots,$

$$\|\mathfrak{U}^n\| \le \frac{1}{2\pi} \int\limits_{\Gamma^n} |1-\lambda|^n \left\|(\lambda I - \mathfrak{A})^{-1}\right\| |d\lambda| \le CM \int\limits_{\Gamma^n} |1-\lambda|^n |\lambda|^{-1} |d\lambda|. \tag{6}$$

Clearly, Γ^n can be decomposed as follows:

$$\Gamma^n = \Gamma_1^n \cup \Gamma_2^n \cup \Gamma_3, \qquad n = 0, 1, \dots, \tag{7}$$

where

$$\Gamma_1^n = \Sigma_\alpha^\odot \cap \partial\mathcal{D}\big(0; d(n+1)^{-1}\big), \quad \Gamma_2^n = \mathcal{D}\big(0; d(n+1)^{-1}\big)^\odot \cap \mathcal{D}(0; d) \cap \partial\Sigma_\alpha,$$

and Γ_3 consists of all $\lambda \in \partial\Upsilon(a, \alpha)$ that lie on the circle $\partial\mathcal{D}(0; a)$. Further observe that for all $\lambda \in \Gamma_1^n$,

$$|1-\lambda|^n \le C$$

and

$$|\lambda|^{-1} \le C(n+1),$$

consequently, for $n = 0, 1, \dots,$

$$\int\limits_{\Gamma_1^n} |1-\lambda|^n |\lambda|^{-1} |d\lambda| \le C(n+1) \int\limits_{\Gamma_1^n} |d\lambda| \le C. \tag{8}$$

Next, in view of the restriction $\alpha < \pi/2$, we have

$$|1-\lambda|^n \le \exp\big(-c(n+1)|\lambda|\big) \quad \text{for} \quad \lambda \in \Gamma_2^n,$$

which gives for $n = 0, 1, \dots,$

$$\int\limits_{\Gamma_2^n} |1-\lambda|^n |\lambda|^{-1} |d\lambda| \le C \int\limits_{d(n+1)^{-1}}^{\infty} \exp\big(-(n+1)cx\big) x^{-1}\, dx = C. \tag{9}$$

Finally, it is easily seen that

$$|\lambda|^{-1} \le C \quad \text{if} \quad \lambda \in \Gamma_3,$$

whence it follows that for all $n = 0, 1, \dots,$ since $a < 1$,

$$\int\limits_{\Gamma_3} |1-\lambda|^n |\lambda|^{-1} |d\lambda| \le Ca^n \le C. \tag{10}$$

Therefore, (3) with $\xi = 0$ follows by combining (6), (7), (8), (9), and (10).

In the case when $\xi > 0$, instead of (5) we use the formula

$$\mathfrak{A}^{\xi}\mathfrak{U}^{n} = (2\pi i)^{-1} \int_{\Gamma} \lambda^{\xi}(1-\lambda)^{n}(\lambda I - \mathfrak{A}_n)^{-1}\, d\lambda, \tag{11}$$

where Γ is a positively oriented contour coinciding with the boundary of the set $\Upsilon(a,\alpha)$ and one takes the principal branch of λ^{ξ}. It is not hard to see then, with the aid of the above argument, that, applying the representation (11) and using further estimates, which are similar to (9) and (10), one can show the claim for $\xi > 0$ as well. □

To prove Theorem 1.2, we first show some quantitative estimates for the procedure of analytic continuation.

Lemma 2.1. *Let $\mathfrak{L} \in \mathcal{B}(X)$ and let $\mu \notin \sigma(\mathfrak{L})$, with*

$$\big\|(\mu I - \mathfrak{L})^{-1}\big\| \le M.$$

Then, for any $\lambda \in \mathbb{C}$ such that $|\lambda - \mu| =: r < M$, we have

$$\big\|(\lambda I - \mathfrak{L})^{-1}\big\| \le \frac{M}{1 - rM}\,.$$

Proof. The claim follows, with the aid of a standard argument (cf. Hille & Phillips [9, Section 5.8]), when inserting

$$\frac{d^n}{d\mu^n}\,(\mu I - \mathfrak{L})^{-1} = (-1)^n n!(\mu I - \mathfrak{L})^{-(n+1)} \quad \text{for} \quad \mu \notin \sigma(\mathfrak{L}),$$

into Taylor's expansion of the resolvent

$$(\lambda I - \mathfrak{L})^{-1} = \sum_{n=0}^{\infty} \frac{(\lambda-\mu)^n}{n!}\,\frac{d^n}{d\mu^n}\,(\mu I - \mathfrak{L})^{-1}.$$

□

The last result can be further used to obtain a resolvent estimate in an important particular case.

Lemma 2.2. *Let $\mathfrak{L} \in \mathcal{B}(X)$ be such that, with some $M > 0$,*

$$\big\|(\mu I - \mathfrak{L})^{-1}\big\| \le M|\lambda|^{-1} \quad \textit{for} \quad \operatorname{Re}\lambda \le 0,$$

and let φ be an arbitrary number such that $\pi/2 - \arcsin M^{-1} < \varphi < \pi/2$. Then for each $\lambda \in \Sigma_{\varphi}^{\odot}$, we have

$$\big\|(\lambda I - \mathfrak{L})^{-1}\big\| \le \frac{M}{1 - M\cos\varphi}\,|\lambda|^{-1}.$$

Proof. Let $\lambda \in \mathbb{C}$ be arbitrary, with $\arg\lambda = \varphi$. Then, for $\mu := \dfrac{i\,|\lambda|}{\sin\varphi}$, we have $|\mu - \lambda| = |\mu|\cos\varphi$. With this in mind, using the accepted restriction on $\big\|(\mu I - \mathfrak{L})^{-1}\big\|$ and applying further Lemma 2.1, we obtain, since $|\mu| \ge |\lambda|$,

$$\big\|(\lambda I - \mathfrak{L})^{-1}\big\| \le \frac{M}{1 - M|\mu|^{-1}|\mu-\lambda|}\,|\mu|^{-1} \le \frac{M}{1 - M\cos\varphi}\,|\lambda|^{-1}.$$

By symmetry, the same estimate holds as well for all λ with $\arg\lambda = -\varphi_1$. It is also clear, using the above reasoning, that this result remains valid for all λ such that $\big|\arg(-\lambda)\big| \le \pi - \varphi$. □

Proof of Theorem 1.2 in the case $\xi = 0$. We put $\varphi := \arcsin \dfrac{1}{2M}$, for short.

Assume first that $(n+1)^{-1} \le 2\sin\varphi$. Let then $\Gamma^{n,M}$ be a positively oriented contour given by

$$\Gamma^{n,M} = \Gamma_1^{n,M} \cup \Gamma_2^{n,M} \cup \Gamma_3^{M},$$

where

$$\Gamma_1^{n,M} = \partial\mathcal{D}\big(0;(n+1)^{-1}\big) \cap \Sigma^{\odot}_{\pi/2-\varphi},$$
$$\Gamma_2^{n,M} = \mathcal{D}\big(0;(n+1)^{-1}\big)^{\odot} \cap \mathcal{D}(0;2\sin\varphi) \cap \partial\Sigma_{\pi/2-\varphi},$$
$$\Gamma_3^{M} \ = \partial\mathcal{D}(1;1) \cap \Sigma_{\pi/2-\varphi}.$$

By the accepted conditions, applying Lemma 2.2 yields for $\lambda \in \Sigma^{\odot}_{\pi/2-\varphi}$,

$$\big\|(\lambda I - \mathfrak{A})^{-1}\big\| \le \frac{M}{1 - M\sin\varphi}\,|\lambda|^{-1} = 2M|\lambda|^{-1}, \tag{12}$$

which, in particular, shows that

$$\big\|(\lambda I - \mathfrak{A})^{-1}\big\| \le 2M(n+1) \quad \text{for} \quad \lambda \in \Gamma_1^n. \tag{13}$$

Also, for further reference, observe that, by the accepted conditions,

$$\big\|(\lambda I - \mathfrak{A})^{-1}\big\| \le \frac{M}{2\cos(\arg\lambda)} \quad \text{for all} \quad \lambda \in \partial\mathcal{D}(1;1). \tag{14}$$

Using now formula (5) with $\Gamma^{n,M}$ in place of Γ^n, we have for $n = 0, 1, \ldots,$

$$\begin{aligned}\mathfrak{U}^n = {} & (2\pi i)^{-1} \int\limits_{\Gamma^{n,M}} (1-\lambda)^n (\lambda I - \mathfrak{A})^{-1}\, d\lambda = (2\pi i)^{-1}\Big(\int\limits_{\Gamma_1^{n,M}} \ldots + \int\limits_{\Gamma_2^{n,M}} \ldots + \int\limits_{\Gamma_3^{M}} \ldots \Big) \\ & =: (2\pi i)^{-1}(I_1 + I_2 + I_3).\end{aligned} \tag{15}$$

Since $|1-\lambda|^n \le C$ for $\lambda \in \Gamma_1^{n,M}$, with the aid of (13), it immediately follows that

$$\|I_1\| \le 2M(n+1) \int\limits_{\Gamma_1^{n,M}} |1-\lambda|^n |d\lambda| \le CM, \quad n = 0, 1, \ldots . \tag{16}$$

For further success we note that, by the evident estimates, for $n = 0, 1, \ldots,$

$$\Big|1 - x\exp\Big(i\Big(\frac{\pi}{2} - \varphi\Big)\Big)\Big|^n \le (1 - x\sin\varphi)^{n/2} \le C\exp\Big(-x\,\frac{n+1}{2}\,\sin\varphi\Big), \tag{17}$$

if $0 \le x \le \sin\varphi$, and

$$\Big|1 - x\exp\Big(i\Big(\frac{\pi}{2} - \varphi\Big)\Big)\Big|^n \le 1 \quad \text{if} \quad \sin\varphi \le x \le 2\sin\varphi,$$

a simple calculation yields

$$\int\limits_{1/(n+1)}^{2\sin\varphi} \Big|1 - x\exp\Big(i\Big(\frac{\pi}{2} - \varphi\Big)\Big)\Big|^n \frac{dx}{x} \le C \int\limits_{1/(n+1)}^{\sin\varphi} \exp\Big(-x\,\frac{n+1}{2}\,\sin\varphi\Big)\,\frac{dx}{x} + \int\limits_{\sin\varphi}^{2\sin\varphi} \frac{dx}{x} \le C\log(1+M).$$

Using this and (12), we get for $n = 0, 1, \ldots,$

$$\|I_2\| \le 4M \int\limits_{(n+1)^{-1}}^{2\sin\varphi} \Big|1 - x\exp\Big(i\Big(\frac{\pi}{2} - \varphi\Big)\Big)\Big|^n \frac{dx}{x} \le CM\log(1+M). \tag{18}$$

Finally, since $\left|\dfrac{d\big(2\cos\vartheta\exp(i\vartheta)\big)}{d\vartheta}\right| = 2$ and $|1-\lambda| = 1$ for $\lambda \in \Gamma_3^M$, we find for $n = 0, 1, \ldots,$ with the aid of (14),

$$\|I_3\| \le \frac{M}{2} \int\limits_{\Gamma_3^M} \frac{|d\lambda|}{\cos(\arg\lambda)} \le 2M \int\limits_0^{\pi/2-\varphi} (\cos\vartheta)^{-1} d\vartheta \le CM\log(1+M). \tag{19}$$

Now it follows from (15), (16), (18), and (19) that, if $(n+1)^{-1} \le 2\sin\varphi$,

$$\|\mathfrak{U}^n\| \le CM\log(1+M), \qquad n = 0,1,\dots. \tag{20}$$

A similar argument in the case when $(n+1)^{-1} \ge 2\sin\varphi$ (in this case the path of integration will contain no line segments) gives for $n = 0,1,\dots$,

$$\|\mathfrak{U}^n\| \le CM\log(n+2) \le CM\log(1+M). \tag{21}$$

The claim thus follows by combining (20) and (21). □

Proof of Theorem 1.2 in the case $\xi > 0$. Let, as above, $\varphi = \arcsin\dfrac{1}{2M}$ and let $\widetilde\Gamma^M$ be a positively oriented contour given by

$$\widetilde\Gamma^M = \widetilde\Gamma_1^M \cup \widetilde\Gamma_2^M \cup \widetilde\Gamma_3^M,$$

where

$$\widetilde\Gamma_1^M = \mathcal{D}(0;\sin\varphi) \cap \partial\Sigma_{\pi/2-\varphi},$$
$$\widetilde\Gamma_2^M = \mathcal{D}(0;\sin\varphi)^{\odot} \cap \mathcal{D}\Big(0;\Big(2-\frac{1}{M}\Big)\sin\varphi\Big) \cap \partial\Sigma_{\pi/2-\varphi},$$
$$\widetilde\Gamma_3^M = \Big\{\lambda : \lambda = \Big(2-\frac{1}{M}\Big)\cos(\arg\lambda)\exp(i\arg\lambda)\Big\} \cap \Sigma_{\pi/2-\varphi}.$$

With the aid of (11) and with $\widetilde\Gamma^M$ in place of Γ, we have for $n = 0,1,\dots$,

$$\begin{aligned}\mathfrak{A}^\xi\mathfrak{U}^n = {} & (2\pi i)^{-1}\int\limits_{\widetilde\Gamma^M} \lambda^\xi(1-\lambda)^n(\lambda I-\mathfrak{A})^{-1}\,d\lambda = (2\pi i)^{-1}\Big(\int\limits_{\widetilde\Gamma_1^M}\dots+\int\limits_{\widetilde\Gamma_2^M}\dots+\int\limits_{\widetilde\Gamma_3^M}\dots\Big)\\ =: {} & (2\pi i)^{-1}(J_1+J_2+J_3).\end{aligned} \tag{22}$$

What we do next is that we show suitable estimates for J_1, J_2, and J_3.

A simple estimation, which is similar to that leading to (18), gives for $n = 0,1,\dots$, with the aid of (17),

$$\begin{aligned}\|J_1\| &\le 4M\int\limits_0^{\sin\varphi} x^{\xi-1}\Big|1-x\exp\Big(i\Big(\frac{\pi}{2}-\varphi\Big)\Big)\Big|^n dx \le CM\int\limits_0^{\infty} x^{\xi-1}\exp\Big(-\frac{x}{2}(n+1)\sin\varphi\Big)\,dx\\ &\le CM^{1+\xi}(n+1)^{-\xi}.\end{aligned} \tag{23}$$

Similarly, with the aid of the estimate

$$\Big|1-x\exp\Big(i\Big(\frac{\pi}{2}-\varphi\Big)\Big)\Big| \le \exp\Big(-\sin\varphi\Big(\sin\varphi-\frac{x}{2}\Big)\Big) \quad \text{for} \quad \sin\varphi \le x \le 2\sin\varphi,$$

we obtain in the case $0 < \xi \le 1$, for $n = 0,1,\dots$, since $x^{-1}(1-e^{-x}) \le x^{-\xi}$ for $x > 0$,

$$\begin{aligned}\|J_2\| &\le 4M\int\limits_{\sin\varphi}^{(2-1/M)\sin\varphi} x^{\xi-1}\Big|1-x\exp\Big(i\Big(\frac{\pi}{2}-\varphi\Big)\Big)\Big|^n dx\\ &\le CM(\sin\varphi)^{\xi-1}\int\limits_{\sin\varphi}^{2\sin\varphi}\exp\Big(-(n+1)\sin\varphi\Big(\sin\varphi-\frac{x}{2}\Big)\Big)\,dx\\ &\le CM(\sin\varphi)^{\xi-2}(n+1)^{-1}\Big(1-\exp\Big(-\frac{n+1}{2}\sin^2\varphi\Big)\Big) \le CM^{1+\xi}(n+1)^{-\xi}.\end{aligned} \tag{24}$$

Next, with the aid of (14), using the analytic continuation of the resolvent of $\mathfrak{A}$ from the set $\partial D(1;1)\setminus\{0\}$ towards the origin, within the sector $\Sigma_{\pi/2-\varphi}$, and applying Lemma 2.1, we come to the following estimate for all $\lambda \in \widetilde\Gamma_3^M$:

$$\big\|(\lambda I-\mathfrak{A})^{-1}\big\| \le \frac{M\big(2\cos(\arg\lambda)\big)^{-1}}{1-M\big(2\cos(\arg\lambda)\big)^{-1}M^{-1}\cos(\arg\lambda)} = \frac{M}{\cos(\arg\lambda)}. \tag{25}$$

At the same time we have for $\lambda \in \widetilde{\Gamma}_3^M$,

$$|1-\lambda|^n = \big(1-(2-M^{-1})M^{-1}\cos^2\vartheta\big)^{n/2} \le C\exp\Big(-\frac{n+1}{2M}\cos^2\vartheta\Big). \tag{26}$$

Therefore, since $\left|\dfrac{d\big((2-M^{-1})\cos\vartheta\exp(i\vartheta)\big)}{d\vartheta}\right| \le 2$, using (25), (26), the estimate

$$\int\limits_{\sin\varphi}^{\infty} x^{\xi-1}\exp\Big(-\frac{n+1}{2M}x^2\Big)\,dx \le \int\limits_{\sin\varphi}^{\infty} x^{\xi-1}\exp\Big(-x\,\frac{n+1}{2M}\sin\varphi\Big)\,dx \le C\Big(\frac{M^2}{n+1}\Big)^{\xi},$$

and the above argument, we obtain for $n = 0, 1, \dots,$

$$\begin{aligned}\|J_3\| &\le M\int\limits_{\widetilde{\Gamma}_3^M}|\lambda|^{\xi}\big(\cos(\arg\lambda)\big)^{-1}|1-\lambda|^n\,|d\lambda| \le CM\int\limits_0^{\pi/2-\varphi}(\cos\vartheta)^{\xi-1}\exp\Big(-\frac{n+1}{2M}\cos^2\vartheta\Big)\,d\vartheta\\ &\le CM\Big(\int\limits_{\sin\varphi}^{\infty}x^{\xi-1}\exp\Big(-\frac{n+1}{2M}x^2\Big)\,dx + \exp\Big(-\frac{n+1}{4M}\Big)\Big)\\ &\le CM\Big(\Big(\frac{M^2}{n+1}\Big)^{\xi} + \exp\Big(-\frac{n+1}{4M}\Big)\Big) \le CM^{1+2\xi}(n+1)^{-\xi}.\end{aligned} \tag{27}$$

Combining now (22), (23), (24), and (27), we obtain in the case $0 < \xi \le 1$,

$$\|\mathfrak{A}^{\xi}\mathfrak{U}^n\| \le CM^{1+2\xi}(n+1)^{-\xi} \quad \text{for} \quad n = 0, \dots, \tag{28}$$

which implies the desired result in the case $0 < \xi \le \frac{1}{2}$.

For below reference we also give a slightly different estimate for $\|J_3\|$. More precisely, using the first three lines in (27), we get in the case when $n+1 \le M$,

$$\|J_3\| \le CM\Big(\Big(\frac{M}{n+1}\Big)^{\xi/2}\int\limits_0^{\infty}x^{\xi-1}\exp(-x^2)\,dx + \exp\Big(-\frac{n+1}{4M}\Big)\Big) \le CM^{1+\xi}(n+1)^{-\xi}. \tag{29}$$

Now we derive one more estimate for $\|J_2+J_3\|$, which will be an important ingredient for showing (4) for $\frac{1}{2} < \xi \le \frac{3}{2}$. In fact, putting

$$\lambda_1(\varphi) := \sin\varphi\,\exp\Big(-i\Big(\frac{\pi}{2}-\varphi\Big)\Big),$$

$$\lambda_2(\varphi) := \sin\varphi\,\exp\Big(i\Big(\frac{\pi}{2}-\varphi\Big)\Big),$$

and

$$H(\lambda) := (n+1)^{-1}(1-\lambda)^{n+1}\lambda^{\xi}(\lambda I-\mathfrak{A})^{-1},$$

and integrating by parts, we find

$$\begin{aligned}&\int\limits_{\widetilde{\Gamma}_2^M\cup\widetilde{\Gamma}_3^M}\lambda^{\xi}(1-\lambda)^n(\lambda I-\mathfrak{A})^{-1}\,d\lambda\\ &\qquad= \Big(-H\big(\lambda_2(\varphi)\big)+H\big(\lambda_1(\varphi)\big)\Big) + \int\limits_{\widetilde{\Gamma}_2^M}\big(\xi\lambda^{-1}H(\lambda)-H(\lambda)\,(\lambda I-\mathfrak{A})^{-1}\big)\,d\lambda\\ &\qquad\quad+\int\limits_{\widetilde{\Gamma}_3^M}\big(\xi\lambda^{-1}H(\lambda)-H(\lambda)\,(\lambda I-\mathfrak{A})^{-1}\big)\,d\lambda =: H_1+H_2+H_3.\end{aligned} \tag{30}$$

With the aid of (12) and (17), a direct estimation gives, since $x^{\xi-1}e^{-x} \le C$ for all $x \ge 0$, in the case when $\xi \ge 1$,

$$\|H_1\| \le \begin{cases} CM^{2-\xi}(n+1)^{-1} & \text{if} \quad 0 < \xi < 1, \\ CM^{\xi}(n+1)^{-\xi} & \text{if} \quad \xi \ge 1. \end{cases} \tag{31}$$

Next, using the argument leading to (24) and (27), we get

$$\|H_2\| \le \begin{cases} CM^{3-\xi}(n+1)^{-1} & \text{if} \quad 0 < \xi < 1, \\ CM^{1+\xi}(n+1)^{-\xi} & \text{if} \quad 1 \le \xi \le 2, \end{cases} \tag{32}$$

and

$$\|H_3\| \le \begin{cases} CM^{3-\xi}(n+1)^{-1} & \text{if} \quad 0 < \xi < 1, \\ CM^{2}\log(1+M)\,(n+1)^{-1} & \text{if} \quad \xi = 1, \\ CM^{2\xi}(n+1)^{-\xi} & \text{if} \quad \xi > 1. \end{cases} \tag{33}$$

Clearly, in the case $0 < \xi < 1$, it follows from (31), (32), and (33) that

$$\|H_1 + H_2 + H_3\| \le CM^2(n+1)^{-\xi} \quad \text{if} \quad M \le n+1.$$

Combining this with (22), (23), (24), (29), and (30) proves (4) for $\dfrac{1}{2} < \xi < 1$.

It is also seen that, in the case $1 \le \xi \le \dfrac{3}{2}$, the claim follows by combining (22), (23), (31), (32), and (33).

Further, to show the result for $\dfrac{3}{2} < \xi \le \dfrac{5}{2}$, we integrate again by parts on the right-hand side of (30) and use (31) and the above argument. Note that, similarly to (29), one can find for $\xi > 1$,

$$\|H_3\| \le CM^{(3+\xi)/2}(n+1)^{-(1+\xi)/2},$$

which will be needed at this stage of the proof.

Clearly, using repeatedly the above reasoning, the claim can be shown for any fixed $\xi > 0$ in a finite number of steps. □

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